

# EXACTLY $m$ -COLOURED COMPLETE INFINITE SUBGRAPHS

BHARGAV P. NARAYANAN

**ABSTRACT.** We say a graph is exactly  $m$ -coloured if we have a surjective map from the edges to some set of  $m$  colours. The question of finding exactly  $m$ -coloured complete subgraphs was first considered by Erickson in 1994; in 1999, Stacey and Weidl partially settled a conjecture made by Erickson and raised some further questions. In this paper, we shall study, for a colouring of the edges of the complete graph on  $\mathbb{N}$  with exactly  $k$  colours, how small the set of natural numbers  $m$  for which there exists an exactly  $m$ -coloured complete infinite subgraph can be. We prove that this set must have size at least  $\sqrt{2k}$ ; this bound is tight for infinitely many values of  $k$ . We also obtain a version of this result for colourings that use infinitely many colours.

## 1. INTRODUCTION

A classical result of Ramsey [10] says that when the edges of a complete graph on a countably infinite vertex set are finitely coloured, one can always find a complete infinite subgraph all of whose edges have the same colour.

Ramsey's Theorem has since been generalised in many ways; most of these generalisations are concerned with finding other monochromatic structures. For a survey of many of these generalisations, see the book of Graham, Rothschild and Spencer [8]. Ramsey theory has witnessed many developments over the last fifty years and continues to be an area of active research today; see for example [9], [1], [13], [2].

Alternatively, anti-Ramsey theory, which originates in a paper of Erdős, Simonovits and Sós [5], is concerned with finding large “rainbow coloured” or “totally multicoloured” structures. Between these two ends of the spectrum, one could consider the question of finding structures which are coloured with exactly  $m$  different colours as was first done by Erickson [6]; this is the line of enquiry that we pursue here.

## 2. OUR RESULTS

For a set  $X$ , denote by  $X^{(2)}$  the set of all unordered pairs of elements of  $X$ ; equivalently,  $X^{(2)}$  is the complete graph on the vertex set  $X$ . As always,  $[n]$  will denote  $\{1, \dots, n\}$ , the set of the

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first  $n$  natural numbers. By a colouring of a graph  $G$ , we will always mean a colouring of the edges of  $G$ .

Let  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  be a surjective  $k$ -colouring of the edges of the complete graph on the natural numbers with  $k \geq 2$  colours. We say that a subset  $X \subset \mathbb{N}$  is (*exactly*)  $m$ -coloured if  $\Delta(X^{(2)})$ , the set of values attained by  $\Delta$  on the edges with both endpoints in  $X$ , has size exactly  $m$ . Our aim in this paper is to study the set

$$\mathcal{F}_\Delta := \{m \in [k] : \exists X \subset \mathbb{N} \text{ such that } X \text{ is infinite and } m\text{-coloured}\}.$$

Clearly,  $k \in \mathcal{F}_\Delta$  as  $\Delta$  is surjective. Ramsey's theorem tells us that  $1 \in \mathcal{F}_\Delta$ . Furthermore, Erickson [6] noted that a fairly straightforward application of Ramsey's theorem enables one to show that  $2 \in \mathcal{F}_\Delta$  for any surjective  $k$ -colouring  $\Delta$  with  $k \geq 2$ . He also conjectured that with the exception of 1, 2 and  $k$ , no other elements are guaranteed to be in  $\mathcal{F}_\Delta$  and that if  $k > k' > 2$ , then there is a surjective  $k$ -colouring  $\Delta$  such that  $k' \notin \mathcal{F}_\Delta$ . Stacey and Weidl [11] settled this conjecture in the case where  $k$  is much bigger than  $k'$ . More precisely, for any  $k' > 2$ , they showed that there is a constant  $C_{k'}$  such that if  $k > C_{k'}$ , then there is a surjective  $k$ -colouring  $\Delta$  such that  $k' \notin \mathcal{F}_\Delta$ .

In this note, we shall be interested in the set of possible sizes of  $\mathcal{F}_\Delta$ . Since  $\mathcal{F}_\Delta \subset [k]$ , we have  $|\mathcal{F}_\Delta| \leq k$  and it is easy to see that equality is in fact possible. Things are not so clear when we turn to the question of lower bounds. Let us define

$$\psi(k) := \min_{\Delta: \mathbb{N}^{(2)} \rightarrow [k]} |\mathcal{F}_\Delta|.$$

We are able to prove the following lower bound for  $\psi(k)$ .

**Theorem 1.** *Let  $n \geq 2$  be the largest natural number such that  $k \geq \binom{n}{2} + 1$ . Then  $\psi(k) \geq n$ .*

It is not hard to check that Theorem 1 is tight when  $k = \binom{n}{2} + 1$  for some  $n \geq 2$ . To this end, we consider the “small-rainbow colouring”  $\Delta$  which colours all the edges with both endpoints in  $[n]$  with  $\binom{n}{2}$  distinct colours and all the remaining edges with the one colour that has not been used so far. Clearly  $|\mathcal{F}_\Delta| = n$ , and so Theorem 1 is best possible for infinitely many values of  $k$ .

Turning to the question of upper bounds for  $\psi$ , the small-rainbow colouring demonstrates that  $\psi(k) = O(\sqrt{k})$  for infinitely many values of  $k$ . However, the obvious generalisations of the small-rainbow colouring described above fail to give us any good upper bounds on  $\psi(k)$  for general  $k$ ; in particular, we are unable to decide if  $\psi(k) = o(k)$  for all  $k \in \mathbb{N}$ . However, by considering colourings that colour all the edges of a small complete bipartite graph with distinct colours (as opposed to a small complete graph) and making use of some number theoretic estimates of Tenenbaum [12] and Ford [7], we get reasonably close to such a statement.

**Theorem 2.** *There exists a subset  $A$  of the natural numbers of asymptotic density one such that for all  $k \in A$ , we have*

$$\psi(k) = O\left(\frac{k}{(\log \log k)^\delta (\log \log \log k)^{3/2}}\right)$$

where  $\delta = 1 - \frac{1+\log \log 2}{\log 2} \approx 0.086 > 0$ .

Canonical Ramsey theory, which originates in a paper of Erdős and Rado [4], involves the study of colourings that use infinitely many colours. We are able to prove a result similar to Theorem 1 for such colourings.

We say that  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  is an *infinite colouring* if it is a colouring that uses infinitely many colours; in other words, if the image of  $\Delta$  is infinite. For an infinite colouring  $\Delta$ , the analogue of  $\mathcal{F}_\Delta$  that is of interest to us is the set

$$\mathcal{G}_\Delta := \{m \in \mathbb{N} : \exists X \subset \mathbb{N} \text{ such that } X \text{ is } m\text{-coloured}\}.$$

The difference between  $\mathcal{G}_\Delta$  and  $\mathcal{F}_\Delta$  is that we also consider finite complete subgraphs in the definition of  $\mathcal{G}_\Delta$ . Since the set of colours is no longer finite, it might be the case that for each infinite subset  $X$  of  $\mathbb{N}$ ,  $\Delta(X^{(2)})$  is infinite; this motivates our definition.

By considering the unique injective colouring  $\Delta$  that colours each edge of the complete graph on  $\mathbb{N}$  with a distinct colour, we see that unless  $m = \binom{n}{2}$  for some  $n \geq 2$ ,  $m$  is not guaranteed to be a member of  $\mathcal{G}_\Delta$ . In the other direction, since an edge is a 1-coloured complete graph,  $\binom{2}{2} = 1$  is always an element of  $\mathcal{G}_\Delta$ . With a little work, one can prove that  $\binom{3}{2} = 3$  is also always an element of  $\mathcal{G}_\Delta$ . But for  $n \geq 4$ , we are unable to decide whether or not there exists a colouring  $\Delta$  with infinitely many colours such that  $\binom{n}{2} \notin \mathcal{G}_\Delta$ .

However, we can prove the following analogue of Theorem 1 for infinite colourings.

**Theorem 3.** *Let  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  be an infinite colouring and suppose  $n \geq 2$  is a natural number. Then,  $|\mathcal{G}_\Delta \cap [\binom{n}{2}]| \geq n - 1$ .*

Again, by considering the injective colouring that colours each edge with a distinct colour, it is easy to see that Theorem 3 is best possible.

The paper is organised as follows. In the next section, we prove our lower bounds, namely Theorems 1 and 3. We remark that we do not prove Theorem 1 and 3 as stated. Instead, we prove two stronger structural results that in turn imply the theorems. We postpone the statements of these results since they depend on a certain notion of homogeneity that we will introduce in the next section. In Section 3, we describe how Theorem 2 follows from certain divisor estimates. We conclude by mentioning some open problems in Section 4.

### 3. LOWER BOUNDS

In this section, we prove Theorem 1 by proving a stronger structural result, namely Theorem 6. The proof of Theorem 3 via Theorem 8 is very similar and we shall only highlight the main differences in the proofs.

We first introduce a notational convenience. Given a colouring  $\Delta$  of  $\mathbb{N}^{(2)}$ , a vertex  $v \in \mathbb{N}$ , and a subset  $X \subset \mathbb{N} \setminus \{v\}$ , we say that a colour  $c$  is a *new colour from  $v$  into  $X$*  if some edge from  $v$  to  $X$  is coloured  $c$  by  $\Delta$  and also, no edge in  $X^{(2)}$  is coloured  $c$  by  $\Delta$ . We write  $N_\Delta(v, X)$  for the set of new colours from  $v$  into  $X$ .

**3.1. Proof of Theorem 1.** Before we prove Theorem 1, we note that Erickson's argument showing that  $2 \in \mathcal{F}_\Delta$  can be generalised to give a quick proof of  $\psi(k) = \Omega(\log k)$ .

**Lemma 4.** *Let  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  and suppose  $l \in \mathcal{F}_\Delta$  and  $l < k$ . Then there is an  $l' \in \mathcal{F}_\Delta$  such that  $l + 1 \leq l' \leq 2l$ .*

Note that Lemma 4, coupled with the fact that we always have  $1 \in \mathcal{F}_\Delta$ , implies that  $\psi(k) \geq 1 + \log_2 k$ .

*Proof of Lemma 4.* Let  $X \subset \mathbb{N}$  be a maximal  $l$ -coloured set. As  $l < k$ ,  $X \neq \mathbb{N}$ . Pick  $v \in \mathbb{N} \setminus X$ . Note that  $N_\Delta(v, X) \neq \emptyset$  since otherwise  $X \cup \{v\}$  is  $l$ -coloured, which contradicts the maximality of  $X$ .

If  $|N_\Delta(v, X)| \leq l$ , then  $X \cup \{v\}$  is  $l'$ -coloured for some  $l + 1 \leq l' \leq 2l$ . So suppose  $|N_\Delta(v, X)| \geq l + 1$ . By the pigeonhole principle, there is an infinite subset  $X'$  of  $X$  such that all the vertices in  $X'$  are connected to  $v$  by edges of a single colour, say  $c$ .

We consider two cases. If  $c \in N_\Delta(v, X)$ , we pick  $l - 1$  vertices from  $X$  which are joined to  $v$  by edges coloured with  $l - 1$  distinct colours from  $N_\Delta(v, X) \setminus \{c\}$ . If on the other hand  $c \notin N_\Delta(v, X)$ , we pick  $l$  vertices from  $X$  which are joined to  $v$  by edges coloured with  $l$  distinct colours from  $N_\Delta(v, X)$ . Call this set of  $l - 1$  or  $l$  vertices  $X''$ .

In both cases, it is easy to check that  $X' \cup X'' \cup \{v\}$  is  $l'$ -coloured with  $l + 1 \leq l' \leq 2l$ .  $\square$

Consequently, we have the following corollary.

**Corollary 5.** *Let  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  and suppose  $n$  is a natural number such that  $k \geq 2^n + 1$ . Then  $\mathcal{F}_\Delta \cap ([2^{n+1}] \setminus [2^n]) \neq \emptyset$ .*

We shall show that for any  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  with  $k \geq \binom{n}{2} + 1$  for some  $n$ , we can find  $n$  nested subsets  $A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_n$  of  $\mathbb{N}$  such that  $\Delta(A_1^{(2)}) \subsetneq \Delta(A_2^{(2)}) \subsetneq \dots \subsetneq \Delta(A_n^{(2)})$ . To do this, we introduce the notion of  $n$ -homogeneity on which our first structural result, Theorem 6, hinges.

For an ordered  $n$ -tuple  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , write  $\bar{X}_i$  for the set  $X_1 \cup X_2 \dots \cup X_i$ . Given a colouring  $\Delta$ , we call  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , with each  $X_i$  a non-empty subset of  $\mathbb{N}$ ,  $n$ -homogeneous with respect to  $\Delta$  if the following conditions are met:

- (1)  $X_i \cap X_j = \emptyset$  for  $i \neq j$ ,
- (2)  $X_1$  is infinite and 1-coloured,
- (3)  $\Delta(\bar{X}_1^{(2)}) \subsetneq \Delta(\bar{X}_2^{(2)}) \subsetneq \dots \subsetneq \Delta(\bar{X}_n^{(2)})$ ,
- (4) for each  $X_i$  with  $2 \leq i \leq n$ , every  $v \in X_i$  satisfies

$$N_\Delta(v, \bar{X}_{i-1}) = \Delta(\bar{X}_i^{(2)}) \setminus \Delta(\bar{X}_{i-1}^{(2)}),$$

- (5)  $|\Delta(\bar{X}_n^{(2)})| \leq \binom{n}{2} + 1$ .

Rather than proving Theorem 1, we prove the following stronger statement.

**Theorem 6.** *Let  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  and suppose  $n$  is a natural number such that  $k \geq \binom{n}{2} + 1$ . Then there exists an  $n$ -homogeneous tuple with respect to  $\Delta$ .*

Before we prove Theorem 6, let us first recall the lexicographic order on  $\mathbb{N}^r$ : We say that  $(a_1, a_2, \dots, a_r) < (b_1, b_2, \dots, b_r)$  if for some  $r' \leq r - 1$  we have  $a_i = b_i$  for  $1 \leq i \leq r'$  and  $a_{r'+1} < b_{r'+1}$ .

Note that if  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  is  $n$ -homogeneous, then by Condition 4, the set  $N_\Delta(v, \bar{X}_{i-1})$  is identical for all  $v \in X_i$  for  $2 \leq i \leq n$ . For  $n \geq 2$ , define the *rank* of an  $n$ -homogeneous tuple  $\mathbf{X}$  to be the  $(n-1)$ -tuple  $(x_1, x_2, \dots, x_{n-1})$ , where  $x_i$  is the number of new colours from any vertex in  $X_{i+1}$  into the set  $\bar{X}_i$ . Note that the rank of an  $n$ -homogeneous tuple is an  $(n-1)$ -tuple of natural numbers and so we can compare ranks by the lexicographic order on  $\mathbb{N}^{n-1}$ .

*Proof of Theorem 6.* We proceed by induction on  $n$ . The case  $n = 1$  is Ramsey's theorem. Suppose that  $k \geq \binom{n+1}{2} + 1$  and assume inductively that at least one  $n$ -homogeneous tuple exists.

From the set of all  $n$ -homogeneous tuples, pick one with minimal rank in the lexicographic order, say  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ . If  $n = 1$ , the rank is immaterial; it suffices to pick  $\mathbf{X} = (X_1)$  such that  $X_1$  is an infinite 1-coloured set. We will build an  $(n+1)$ -homogeneous tuple from  $\mathbf{X}$ .

Note that  $k \geq \binom{n+1}{2} + 1 > \binom{n}{2} + 1$ . Since  $\Delta$  is surjective and attains at most  $\binom{n}{2} + 1$  different values inside  $\bar{X}_n$ , clearly  $\mathbb{N} \setminus \bar{X}_n \neq \emptyset$ . We consider two cases.

CASE 1:  $N_\Delta(v, \bar{X}_n) \neq \emptyset$  for some  $v \in \mathbb{N} \setminus \bar{X}_n$ .

If  $|N_\Delta(v, \bar{X}_n)| \leq n$ , then it is easy to check that  $(X_1, X_2, \dots, X_n, \{v\})$  is an  $(n+1)$ -homogeneous tuple and we are done. So, assume without loss of generality that  $|N_\Delta(v, \bar{X}_n)| \geq n+1$ .

Let  $j$  be the smallest index such that  $N_\Delta(v, \bar{X}_j) \neq \emptyset$ . Since  $N_\Delta(v, \bar{X}_n) \neq \emptyset$ , this minimal index  $j$  exists. We now build our  $(n+1)$ -homogeneous tuple  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{n+1})$  as follows.

Set  $Y_1 = X_1, Y_2 = X_2, \dots, Y_{j-1} = X_{j-1}$ . We define  $Y_j$  as follows. Choose  $c \in N_\Delta(v, \bar{X}_j)$ . Note that by the minimality of  $j$ ,  $N_\Delta(v, \bar{X}_{j-1}) = \emptyset$  and so all the edges between  $v$  and  $\bar{X}_j$  coloured  $c$  are actually edges between  $v$  and  $X_j$ . Take  $Y_j \subset X_j$  to be that (non-empty) subset of vertices  $v'$  of  $X_j$  such that the edge between  $v$  and  $v'$  is either coloured  $c$  or with a colour from  $\Delta(\bar{X}_j^{(2)})$  (and hence a colour not in  $N_\Delta(v, \bar{X}_j)$ ). Note that if  $j = 1$ , we can always choose  $c$  such that  $Y_1$  is an infinite subset of  $X_1$ .

Next, set  $Y_{j+1} = \{v\}$ . Now, note that the only colour from  $\Delta(\bar{Y}_{j+1}^{(2)})$  that might possibly occur in  $N_\Delta(v, \bar{X}_n)$  is  $c$ . So we can choose  $v_1, v_2, \dots, v_{n-j}$  from  $X_n \cup X_{n-1} \dots \cup X_{j+1} \cup (X_j \setminus Y_j)$  such that these  $n-j$  vertices are joined to  $v$  by edges which are all coloured by distinct elements of  $N_\Delta(v, \bar{X}_n) \setminus \{c\}$ . Set  $Y_{j+2} = \{v_1\}, Y_{j+3} = \{v_2\}, \dots, Y_{n+1} = \{v_{n-j}\}$ .

We claim that  $\mathbf{Y}$  is an  $(n+1)$ -homogeneous tuple. Indeed, Conditions 1 and 2 are obviously satisfied.

To check Condition 3, first note that  $\Delta(\bar{Y}_1^{(2)}) \subsetneq \Delta(\bar{Y}_2^{(2)}) \subsetneq \dots \subsetneq \Delta(\bar{Y}_j^{(2)})$  follows from the  $n$ -homogeneity of  $\mathbf{X}$ . Next,  $\Delta(\bar{Y}_j^{(2)}) \subsetneq \Delta(\bar{Y}_{j+1}^{(2)})$  since  $v$  is joined to at least one vertex in  $Y_j$  by an edge coloured with  $c$  and we know that  $c$  is a new colour from  $v$  into  $\bar{Y}_j$ . Finally,  $\Delta(\bar{Y}_{j+1}^{(2)}) \subsetneq \Delta(\bar{Y}_{j+2}^{(2)}) \subsetneq \dots \subsetneq \Delta(\bar{Y}_{n+1}^{(2)})$  follows from the choice of  $v_1, v_2, \dots, v_{n-j}$ . So Condition 3 is also satisfied.

Condition 4 for each of  $Y_1, Y_2, \dots, Y_j$  is equivalent to the same condition for  $X_1, X_2, \dots, X_j$  respectively. Furthermore, Condition 4 is also satisfied by each of  $Y_{j+1}, Y_{j+2}, \dots, Y_{n+1}$  since they each contain exactly one vertex.

Finally, we check Condition 5. Clearly,  $\Delta(\bar{Y}_{n+1}^{(2)})$  is a subset of  $\Delta(\bar{X}_n^{(2)}) \cup T$  for some subset  $T$  of  $N_\Delta(v, \bar{X}_n)$  of size at most  $n$ . Hence, we see that  $|\Delta(\bar{Y}_{n+1}^{(2)})| \leq \binom{n+1}{2} + 1 + n = \binom{n+1}{2} + 1$ .

CASE 2:  $N_\Delta(v, \bar{X}_n) = \emptyset$  for every  $v \in \mathbb{N} \setminus \bar{X}_n$ .

It is here that we use the fact that  $\mathbf{X}$  has minimal lexicographic rank. To deal with this case, we need the following lemma.

**Lemma 7.** *Let  $\mathbf{X}$  be an  $n$ -homogeneous tuple of minimal lexicographic rank and suppose  $N_\Delta(v, \bar{X}_n) = \emptyset$  for some  $v \in \mathbb{N} \setminus \bar{X}_n$ . Then there is an  $n$ -homogeneous tuple  $\mathbf{X}'$  such that  $X'_j = X_j \cup \{v\}$  for some  $j \in [n]$ , and  $X'_i = X_i$  for  $1 \leq i \leq n, i \neq j$ .*

*Proof.* If  $N_\Delta(v, \bar{X}_i) = \emptyset$  for  $1 \leq i \leq n$ , then  $(X_1 \cup \{v\}, X_2, \dots, X_n)$  is  $n$ -homogeneous and we have  $\mathbf{X}'$  as required.

Hence, let  $j < n$  be largest index such that  $N_\Delta(v, \bar{X}_j) \neq \emptyset$ . So by the definition of  $j$ ,  $N_\Delta(v, \bar{X}_i) = \emptyset$  for  $j < i \leq n$ . We claim that  $\mathbf{X}' = (X_1, X_2, \dots, X_j, X_{j+1} \cup \{v\}, X_{j+2}, \dots, X_n)$  is  $n$ -homogeneous.

Consider a colour  $c$  that belongs to  $N_\Delta(v, \bar{X}_j)$ . Since  $N_\Delta(v, \bar{X}_{j+1}) = \emptyset$ , this means that  $c$  must occur in  $\Delta(\bar{X}_{j+1}^{(2)}) \setminus \Delta(\bar{X}_j^{(2)})$ . But, by Condition 4, for each  $v' \in X_{j+1}$ ,  $N_\Delta(v', \bar{X}_j) = \Delta(\bar{X}_{j+1}^{(2)}) \setminus \Delta(\bar{X}_j^{(2)})$ . Hence,  $N_\Delta(v, \bar{X}_j) \subset N_\Delta(v', \bar{X}_j)$  for  $v' \in X_{j+1}$ .

Observe that since  $N_\Delta(v, \bar{X}_i) = \emptyset$  for  $j < i \leq n$ , we have  $N_\Delta(v', \bar{X}_{i-1}) = N_\Delta(v', \bar{X}_{i-1} \cup \{v\})$  for each  $v' \in X_i$  with  $j < i \leq n$ . From this, it is easy to see that  $\mathbf{X}'$  is  $n$ -homogeneous if  $N_\Delta(v, \bar{X}_j) = N_\Delta(v', \bar{X}_j)$  for  $v' \in X_{j+1}$ .

So suppose that  $N_\Delta(v, \bar{X}_j) \subsetneq N_\Delta(v', \bar{X}_j)$  for  $v' \in X_{j+1}$ . Consider then the  $n$ -tuple  $\mathbf{Z} = (X_1, X_2, \dots, X_j, \{v\}, X_{j+1}, X_{j+2}, \dots, X_{n-1})$ . We claim that  $\mathbf{Z}$  is  $n$ -homogeneous and has strictly smaller lexicographic rank than  $\mathbf{X}$ , which is a contradiction.

We first check the  $n$ -homogeneity of  $\mathbf{Z}$ . Clearly, Conditions 1 and 2 are satisfied by  $\mathbf{Z}$ .

To check Condition 3, first note that  $\Delta(\bar{Z}_1^{(2)}) \subsetneq \Delta(\bar{Z}_2^{(2)}) \subsetneq \dots \subsetneq \Delta(\bar{Z}_{j+1}^{(2)})$  follows from the  $n$ -homogeneity of  $\mathbf{X}$  and the fact that  $N_\Delta(v, \bar{X}_j) \neq \emptyset$ . Next,  $\Delta(\bar{Z}_{j+1}^{(2)}) \subsetneq \Delta(\bar{Z}_{j+2}^{(2)})$  since  $N_\Delta(v, \bar{X}_j) \subsetneq N_\Delta(v', \bar{X}_j)$  for  $v' \in X_{j+1}$ . Finally,  $\Delta(\bar{Z}_{j+2}^{(2)}) \subsetneq \Delta(\bar{Z}_{j+3}^{(2)}) \subsetneq \dots \subsetneq \Delta(\bar{Z}_n^{(2)})$  since we know that  $N_\Delta(v', \bar{X}_{i-1} \cup \{v\}) = N_\Delta(v', \bar{X}_{i-1}) \neq \emptyset$  for each  $v' \in X_i$  with  $j < i \leq n$ . So  $\mathbf{Z}$  satisfies Condition 3.

Condition 4 is satisfied trivially by each of  $Z_1, Z_2, \dots, Z_j$ . Condition 4 holds for  $Z_{j+1}$  since  $v$  is the only element in  $Z_{j+1}$ . We know that  $N_\Delta(v, \bar{X}_{j+1}) = \emptyset$ . Hence, Condition 4 holds for  $Z_{j+2}$  since for any vertex  $v' \in Z_{j+2} = X_{j+1}$ , we see that  $N_\Delta(v', \bar{Z}_{j+1}) = N_\Delta(v', \bar{X}_j) \setminus N_\Delta(v, \bar{X}_j) = \Delta(\bar{Z}_{j+2}^{(2)}) \setminus \Delta(\bar{Z}_{j+1}^{(2)})$ . Finally, Condition 4 holds for each  $Z_i$  with  $j+2 < i \leq n$  by the fact that  $N_\Delta(v', \bar{X}_{i-1} \cup \{v\}) = N_\Delta(v', \bar{X}_{i-1})$  for each  $v' \in X_i$ . It is easy to see that Condition 5 holds since  $N_\Delta(v, \bar{X}_n) = \emptyset$ .

That  $\mathbf{Z}$  has smaller lexicographic rank than  $\mathbf{X}$  is clear from the fact that  $N_\Delta(v, \bar{X}_j) \subsetneq N_\Delta(v', \bar{X}_j)$  for  $v' \in X_{j+1}$ . This completes the proof of the lemma.  $\square$

We have assumed that  $N_\Delta(v, \bar{X}_n) = \emptyset$  for each  $v \in \mathbb{N} \setminus \bar{X}_n$ . Now,  $\Delta$  is surjective, so there must exist two vertices  $v_1$  and  $v_2$  in  $\mathbb{N} \setminus \bar{X}_n$  such that edge joining  $v_1$  and  $v_2$  is coloured with a colour  $c$  not in  $\Delta(\bar{X}_n^{(2)})$ .

Let  $\mathbf{X}'$  be the  $n$ -homogeneous tuple that we get by applying Lemma 7 to  $\mathbf{X}$  and  $v_1$ . Then, clearly  $N_\Delta(v_2, \bar{X}'_n) = \{c\}$ . Thus,  $(X'_1, X'_2, \dots, X'_n, \{v_2\})$  is an  $(n+1)$ -homogeneous tuple. This completes the proof of the theorem.  $\square$

**3.2. Proof of Theorem 3.** As we mentioned earlier, the proof of Theorem 3 is very similar to that of Theorem 1 and, also goes via a stronger structural result. We only highlight the main differences.

To prove Theorem 3, we need to alter the definition of  $n$ -homogeneity slightly. We relax Condition 2; instead of demanding that our first set  $X_1$  be infinite and 1-coloured, we only require that  $|X_1| = 1$ .

More precisely, given a colouring  $\Delta$ , we call an  $n$ -tuple  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , with each  $X_i$  a non-empty subset of  $\mathbb{N}$ , *weakly homogeneous with respect to  $\Delta$*  if the following conditions are met:

- (1)  $X_i \cap X_j = \emptyset$  for  $i \neq j$ ,
- (2)  $|X_1| = 1$ ,
- (3)  $\emptyset = \Delta(\bar{X}_1^{(2)}) \subsetneq \Delta(\bar{X}_2^{(2)}) \subsetneq \dots \subsetneq \Delta(\bar{X}_n^{(2)})$ ,
- (4) for each  $X_i$  with  $2 \leq i \leq n$ , every  $v \in X_i$  satisfies

$$N_\Delta(v, \bar{X}_{i-1}) = \Delta(\bar{X}_i^{(2)}) \setminus \Delta(\bar{X}_{i-1}^{(2)}),$$

- (5)  $|\Delta(\bar{X}_n^{(2)})| \leq \binom{n}{2}$ .

Theorem 3 is an easy consequence of the following stronger statement.

**Theorem 8.** *Let  $\Delta : \mathbb{N}^{(2)} \rightarrow \mathbb{N}$  be an infinite colouring and suppose  $n \geq 2$  is a natural number. Then there exists an  $n$ -weakly homogeneous tuple with respect to  $\Delta$ .*

The proof is essentially identical to that of Theorem 6. Note that we only use the finiteness of the set of colours in two places in the proof of Theorem 6. First, to produce an infinite 1-coloured set for the base case of the induction and second, to ensure that the subset  $Y_1$  of  $X_1$  that we construct in the inductive step (in Case 1) is infinite. The definition of weak homogeneity gets around both these difficulties.

#### 4. UPPER BOUNDS

Erdős proved in [3] that for a natural number  $n$ , the set  $P_n = \{ab : a, b \leq n\}$  has size  $o(n^2)$ . We base the proof of Theorem 2 on the observation that  $P_n$  is exactly the set of sizes of all induced subgraphs of a complete bipartite graph between two equal vertex classes of size  $n$ .

Let  $H(x, y, z)$  be the number of natural numbers  $n \leq x$  having a divisor in the interval  $(y, z]$ . Tenenbaum [12] showed that

$$(4.1) \quad H(x, y, z) = (1 + o(1))x \text{ if } \log y = o(\log z), z \leq \sqrt{x}.$$

Ford [7] proved that we have

$$(4.2) \quad H(x, y, 2y) = \Theta\left(\frac{x}{(\log y)^\delta (\log \log y)^{3/2}}\right) \text{ if } 3 \leq y \leq \sqrt{x},$$

where  $\delta = 1 - \frac{1 + \log \log 2}{\log 2}$ . Armed with these two facts, we can now prove Theorem 2.

*Proof of Theorem 2.* We shall take

$$A = \{k : \exists a, b \in \mathbb{N} \text{ with } k - 1 = ab \text{ and } \log k \leq a \leq b\}.$$



It follows from (4.1) that  $H(x, \log x, \sqrt{x}) = (1 + o(1))x$ ; as an easy consequence, we have that  $A$  has asymptotic density one. Now, for a fixed  $k \in A$  with  $k - 1 = ab$ , consider a surjective  $k$ -colouring  $\Delta$  of the complete graph on  $\mathbb{N}$  which colours all the edges of the complete bipartite graph between  $[a]$  and  $[b] \setminus [a]$  with  $ab$  distinct colours and all other edges with the one unused colour. It is easy to then see that

$$\mathcal{F}_\Delta = \{a'b' + 1 : a' \leq a, b' \leq b\}.$$

Now, for any element  $a'b' + 1 \in \mathcal{F}_\Delta$ , we have  $a/2^{i+1} < a' \leq a/2^i$  for some  $i \geq 0$  and so  $a'b' \leq ab/2^i$ . Thus,

$$|\mathcal{F}_\Delta| \leq \sum_{i \geq 0} H\left(\frac{ab}{2^i}, \frac{a}{2^{i+1}}, \frac{a}{2^i}\right).$$

Using Ford's estimate (4.2) for  $H(x, y, 2y)$  and the fact that  $a \geq \log k$ , we obtain that

$$\psi(k) = O\left(\frac{k}{(\log \log k)^\delta (\log \log \log k)^{3/2}}\right)$$

for all  $k \in A$ . □

## 5. CONCLUDING REMARKS

Our results raise many questions that we cannot yet answer. We suspect that something much stronger than Corollary 5 is true.

**Conjecture 9.** *Let  $\Delta : \mathbb{N}^{(2)} \rightarrow [k]$  and suppose  $n \geq 2$  is a natural number such that  $k \geq \binom{n}{2} + 2$ . Then  $\mathcal{F}_\Delta \cap \left(\left[\binom{n+1}{2} + 1\right] \setminus \left[\binom{n}{2} + 1\right]\right) \neq \emptyset$ .*

If true, note that this statement would imply Theorem 1. When  $n = 2$ , the conjecture is implied by Corollary 5. We are able to prove the first non-trivial instance of Conjecture 9, namely that when  $k \geq 5$ ,  $\mathcal{F}_\Delta \cap \{5, 6, 7\} \neq \emptyset$ , but the proof we possess sheds no light on how to prove the conjecture in general.

We strongly suspect that the function  $\psi$  is quite far from being monotone. We have shown that  $\psi\left(\binom{n}{2} + 1\right) = n$  and that  $\psi\left(\binom{n+1}{2} + 1\right) = n + 1$ , and it is an easy consequence of our results  $\psi\left(\binom{n}{2} + 2\right) = n + 1$ . It appears to be true that even  $\psi\left(\binom{n}{2} + 3\right)$  is much bigger than  $n$ , though we cannot even prove  $\psi\left(\binom{n}{2} + 3\right) > n + 1$ .

**Conjecture 10.** *There is an absolute constant  $\epsilon > 0$  such that  $\psi\left(\binom{n}{2} + 3\right) > (1 + \epsilon)n$  for all natural numbers  $n \geq 2$ .*

The problem of determining  $\psi$  completely is of course still open. We do not know the answer to even the following question.

**Problem 11.** Is  $\psi(k) = o(k)$  for all  $k \in \mathbb{N}$ ?

If we restrict our attention to colourings which use every colour but one exactly once, we are led to the following question about induced subgraphs, a positive answer to which would immediately imply that  $\psi(k) = o(k)$  for all  $k \in \mathbb{N}$ .

**Problem 12.** Given  $m \in \mathbb{N}$ , let  $G(m)$  be a graph on exactly  $m$  edges for which  $S(m)$ , the set of sizes of all the induced subgraphs of  $G(m)$ , is smallest. Is  $|S(m)| = o(m)$ ?

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DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE

*E-mail address:* `b.p.narayanan@dpms.cam.ac.uk`